Scheme measuring the Wigner function of a field state in traveling waves

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1 Introduction

Theoretical schemes and realizations of experiments on reconstruction of quantum states became hot topics of quantum optics and atomic physics over the last 20 years. The issue may be concerned with atomic or field states. In the first case it usually refers to vibrational states of an atom inside a trap [1], whereas in the second case it may refer to a field-mode, either trapped inside a QED cavity [2] or in traveling waves [4]. However, only a few number of proposals exhibit a strikingly simple data analysis.

In this report, inspired on Ref.[4], we present a simple procedure to reconstruct the quantum states of traveling waves, leading to their Wigner functions. This permits one to consider optical modes in the scenario since, till now, good optical cavities are not available in laboratories as they are in the microwave domain. The present proposal can also be mapped into existing methods for trapped atoms and fields inside a high-Q microwave cavity.

2 Methodology

The apparatus consists of a Mach-Zehnder interferometer (MZI) including an auxiliary Kerr-medium (KM) and a phase shifter (PS) in one arm (*mode b*), plus two additional beam-splitters (BS) in an external arm (*mode c*). The schematical setup is depicted in the Fig.1. The KM couples one of the internal modes of the MZI (mode b) with an external one (*mode a*) where the field to be measured is injected.

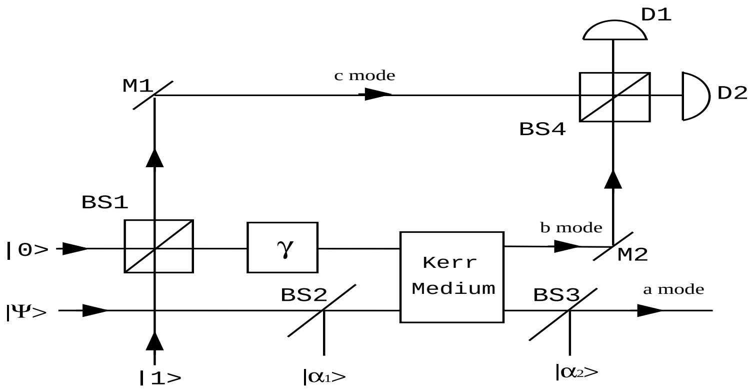

Figure 1: Schematic illustration of the Mach-Zehnder interferometer, including a four-wave mixer, for measuring the state in travelling fields.

Initially, the vacuum and the single photon states enter the modes b and c, respectively, of the first ideal 50/50 symmetric BS of the MZI. The action of he BS1 and BS2 are described by the unitary operator

$$\widehat{R}_{bc} = \exp\left[i\frac{\pi}{4}(\hat{b}^{\dagger}\hat{c} + \hat{b}\hat{c}^{\dagger})\right], \tag{1}$$

where $\hat{b}$ and $\hat{c}$ denote the annihilation operators for the internal modes of the interferometer. Using the Eq.(1) one obtains

$$\widehat{R}_{bc}|0\rangle_b|1\rangle_c = \frac{1}{\sqrt{2}}(\ |0\rangle_b|1\rangle_c + i|1\rangle_b|0\rangle_c\ ), \tag{2}$$

$$\widehat{R}_{bc}|1\rangle_b|0\rangle_c = \frac{1}{\sqrt{2}}(\ |1\rangle_b|0\rangle_c + i|0\rangle_b|1\rangle_c\ ). \tag{3}$$

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The PS is assumed to add a phase $e^{i\gamma}$ to the field crossing it. Thus, just after the BS1 and the PS, the (entangled) state of the whole system reads,

$$|\Psi\rangle_{abc} = \frac{1}{\sqrt{2}}(e^{i\gamma}|0\rangle_b|1\rangle_c + i|1\rangle_b|0\rangle_c)\ |\Psi\rangle_a\ , \tag{4}$$

where $|\Psi\rangle_a$ stands for the initial state entering the mode a, whose Wigner function is to be measured.

In Fig.1, the BS2 and BS3 produce the action of the displacement operator $\hat{D}(\alpha) = exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$ on a quantum state of the field mode a. Thus, after the BS3, the whole state of the system becomes

$$|\Psi'\rangle_{abc} = \frac{1}{\sqrt{2}}(e^{i\gamma}|0\rangle_b|1\rangle_c + i|1\rangle_b|0\rangle_c)\hat{D}(\alpha)\ |\Psi\rangle_a \tag{5}$$

Now, the dispersive Kerr interaction between mode-a and mode-b is described by the interaction Hamiltonian [3]

$$\hat{H}_K = hK\hat{a}^\dagger\hat{a}\hat{b}^\dagger\hat{b} \tag{6}$$

So, the action of the Kerr medium is represented by the unitary operator

$$\hat{U}_K = exp(-i\phi\hat{a}^\dagger\hat{a}\hat{b}^\dagger\hat{b}) \tag{7}$$

where $\phi = Kl/v$, l is the length of the Kerr-medium and v the velocity of light in the medium. Due to the action of the Hamiltonian (6) upon the modes a and b, the entangled state describing the system evolves to,

$$|\Psi''\rangle_{abc} = \frac{1}{\sqrt{2}}\widehat{U}_{\rm K}\left(e^{i\gamma}|0\rangle_b|1\rangle_c + i|1\rangle_b|0\rangle_c\right)\hat{D}(\alpha)|\Psi\rangle_a = \frac{1}{\sqrt{2}}\left(e^{i\gamma}|0\rangle_b|1\rangle_c + i|1\rangle_b|0\rangle_c e^{-i\phi\hat{a}^\dagger\hat{a}}\right)\hat{D}(\alpha)\ |\Psi\rangle_a,$$

since $\widehat{U}_{\rm K}|1\rangle_b = |1\rangle_b e^{-i\phi\hat{a}^\dagger\hat{a}}$ and $\hat{U}_{\rm K}|0\rangle_b = |0\rangle_b$. Now, the action of the BS3 corresponds to a second displacement, leading the Eq.(8) to the following form

$$|\Psi'''\rangle_{abc} = \frac{1}{\sqrt{2}}\left(e^{i\gamma}|0\rangle_b|1\rangle_c\ +\ i|1\rangle_b|0\rangle_c\hat{D}^\dagger(\alpha)e^{-i\phi\hat{a}^\dagger\hat{a}}\hat{D}(\alpha)\right)\ |\Psi\rangle_a\ , \tag{8}$$

and, after the BS4 the foregoing state becomes,

$$|\Psi''''\rangle_{abc} = \frac{1}{2}\left\{|0\rangle_b|1\rangle_c\left[e^{i\gamma} - \hat{D}^\dagger(\alpha)e^{-i\phi\hat{a}^\dagger\hat{a}}\hat{D}(\alpha)\right]\ |\Psi\rangle_a\ +\ i|1\rangle_b|0\rangle_c\left[e^{i\gamma} + \hat{D}^\dagger(\alpha)e^{-i\phi\hat{a}^\dagger\hat{a}}\hat{D}(\alpha)\right]\ |\Psi\rangle_a\right\}. \tag{9}$$

Now, a little algebra shows that

$$\hat{D}^\dagger(\alpha)e^{-i\phi\hat{a}^\dagger\hat{a}}\hat{D}(\alpha)\quad=\quad e^{-i\phi(\hat{a}^\dagger+\alpha^*)(\hat{a}-\alpha)} = e^{-i\phi\hat{a}^\dagger\hat{a}+i(\phi\alpha\hat{a}^\dagger-\phi\alpha^*\hat{a})+i\phi|\alpha|^2}\ . \tag{10}$$

For realistical Kerr-media small values of phase shifts ϕ are easily produced in laboratories; so, when ajusted to high values of α one obtains

$$\hat{D}^\dagger(\alpha)e^{-i\phi\hat{a}^\dagger\hat{a}}\hat{D}(\alpha)\quad\simeq\quad e^{(\beta\hat{a}^\dagger-\phi\beta^*\hat{a})}e^{i\phi|\alpha|^2} = e^{i\phi|\alpha|^2}\hat{D}(\beta), \tag{11}$$

where $\beta = i\phi\alpha$, with $|\beta| = \phi\alpha$ finite. Substituting the Eq.(11) in the Eq.(9) the state of the whole system can be written as

$$|\Psi''''\rangle_{abc}\quad=\quad\frac{e^{i\phi|\alpha|^2}}{2}\left\{|0\rangle_b|1\rangle_c\left[e^{i\xi} - \widehat{D}_a(\beta)\right]\ |\Psi\rangle_a\ +\ i|1\rangle_b|0\rangle_c\left[e^{i\xi} + \widehat{D}_a(\beta)\right]\ |\Psi\rangle_a\right\}, \tag{12}$$

with $\xi = \gamma - \phi|\alpha|^2$.

Note that the state $|\Psi''''\rangle_{abc}$ is also entangled. At this point we can see that if the detector D2 fires the output state $|0\rangle_b|1\rangle_c$ in the BS4 the a-mode is projected onto the state

$$|\Psi\rangle_a^{out} = \frac{e^{i\phi|\alpha|^2}}{2}\left[e^{i\xi} - \widehat{D}_a(\beta)\right]\ |\Psi\rangle_a. \tag{13}$$

The probability $P_{01}(\beta,\xi)$ for the occurrence of this event is given by

$$P_{01}(\beta,\xi) = \frac{1}{4}Tr_a\left\{\left[e^{i\xi} - \widehat{D}_a(\beta)\right]\hat{\rho}_a\left[e^{-i\xi} - \widehat{D}_a^\dagger(\beta)\right]\right\} = \frac{1}{2} - \frac{1}{2}{\rm Re}\left\{e^{-i\xi}Tr_a\left[\widehat{D}(\beta)\hat{\rho}_a\right]\right\}, \tag{14}$$

where $\hat{\rho}_a = |\Psi\rangle_a \langle\Psi|_a$ is the density operator describing the initial (pure) state of the mode a being measured.

Similarly, the probability $P_{10}(\beta,\xi)$ for the occurrence of detection in D1 and no detection in D2 is given by

$$P_{10}(\beta,\xi) = \frac{1}{2} + \frac{1}{2}\mathrm{Re}\left\{e^{-i\xi} Tr_a \left[\widehat{D}(\beta)\hat{\rho}_a\right]\right\}. \tag{15}$$

As this experiment is repeated many times, starting from the same initial field state $\hat{\rho}_a$, one obtains the probibilities $P_{10}(\beta,\xi)$ and $P_{01}(\beta,\xi)$, which furnishes

$$\Delta P(\beta,\xi) = \mathrm{Re}\left\{e^{-i\xi} Tr_a \left[\hat{D}(\beta)\hat{\rho}_a\right]\right\}, \tag{16}$$

where $\Delta P(\beta,\xi) = P_{10}(\beta,\xi) - P_{01}(\beta,\xi)$.

Using the definition of the characteristic function of a state, $\chi(\beta) = Tr\left[\hat{\rho}\widehat{D}(\beta)\right]$, Eq.(16) can be rewritten as

$$\Delta P(\beta,\xi) = \mathrm{Re}\left\{e^{-i\xi}\chi(\beta)\right\}. \tag{17}$$

One sees that the measurement of $\Delta P(\beta,0)$ leads to the real part of the characteristic function, whereas the measurement of $\Delta P(\beta,\pi/2)$ provides its imaginary part. Thus, one obtains

$$\chi(\beta) = \Delta P(\beta,0) + i\Delta P\left(\beta,\frac{\pi}{2}\right). \tag{18}$$

These two measurements actually lead to values of $\chi(\beta)$ at two points (namely, β and $-\beta$) owing to the property $\chi^*(\beta) = \chi(-\beta)$.

Now, since the Wigner function is the Fourier transform of the characteristic function χ, namely,

$$W(z) = \frac{1}{\pi^2}\int d^2\beta\, \chi(\beta) \exp\left(z\beta^* - z^*\beta\right), \tag{19}$$

the determination of $\chi(\beta)$ for a reasonable set of values permits the reconstruction of the Wigner function of the original state entering the external mode a.

3 Conclusion

In summary, this report introduces a method to get the Wigner function via the measurement of the characteristic function describing the state of a traveling field. The present procedure is not restricted to pure states. For mixed states the treatment demands the use of the density-operator formalism.The scheme is available nowadays, being simple and fast concerning with the experiment. The reliability of the scheme is supported by recent technological advances which have achieved photodetectors with efficiency near 100% and stable single-photon sources yielding many single photons into the input port of beam-splitter. Finally, a comparison with the Ref.[4] is pertinent: in [4] the Wigner function is measured directly and in this aspect it is better than ours. However, they require a very large nonlinear suscetibility, yielding $\phi = \pi$, whose experimental implementation is very hard till now. It is worth remembering that our goal is achieved via the use of small values of phase-shifts ϕ, easily available in laboratories using realistic Kerr-media, which constitutes a remarkable result from the experimental point of view.

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