States of the quantized electromagnetic field with highly concentrated phase distribution

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We investigate the generation and properties of a class states having highly peakead phase distribution, describing either a trapped field inside a cavity or a running field. It constitutes a striking example of *partial phase state* and provides a route to a new phase state.

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I. INTRODUCTION

The generation of new states of quantized systems, either for atoms or for trapped and travelling fields, turned out to be an important topic in the last decade. The relevance of new states comes from the properties they may exhibit and their potential applications. Among the several examples one may cite the number state $|n\rangle$ [1], the coherent state $|\alpha\rangle$ [2], the squeezed state $|z,\alpha\rangle$ [3, 4], the phase state $|\theta\rangle$ [5, 6], the Schrödinger's cat state [7], etc. Among various applications one may cite: (i) in fundamentals of physics, as the study of decoherence effects affecting field [8] and atomic states [9]; the study of entangled states and quantum correlations [10]; interference in the phase space and oscillations of the statistical distribution [11]; collapse and reviva l of the atomic inversion when a two-level atom interacts with convenient field states [12]; the use of a field state to determine the properties of another field state [13], etc; (ii) as potential applications: teletransport [14], quantum computation [15], quantum communication [16], quantum cryptography [17], etc.

One of the most basic state of the quantized electromagnetic field is the phase state (**PS**), playing the role of complementary of the number state, in the sense that the number operator $\hat{N}$ and the phase operator $\hat{\phi}$ constitute a conjugate pair of observables [18]: $[\hat{\phi},\hat{N}]=i$. This commutation relation was questioned for a long time after 1963 [19] and the difficulty was then attributed to the lack of good definitions of a phase state and the corresponding (Hermitian) phase operator. So, this placed the phase in a unique position of being a classical observable having no associated Hermitian operator counterpart. This situation was not so unconfortable since most experiments then involved thermal and vacuum fields, for which the phase is not important. However, the advent of the laser and the squeezed light has renewed interest in the problem. Although not being concensual [20], the problem of the PS (and Hermitian phase operator) has been well adressed in 1988, by Pegg and Barnett [5]. It was denoted as $|\theta_m\rangle$ and defined by

$$|\theta_m\rangle = \frac{1}{\sqrt{M+1}}\sum_{n=0}^{M} e^{in\theta_m}\,|n\rangle \tag{1}$$

with $\theta_m = \theta_0 + \left(\frac{2\pi}{M+1}\right)m$, $m = 0,1,2,...,M$, and the corresponding Hermitian phase operator

$$\hat{\phi}_\theta = \sum_{m=0}^{M} \theta_m|\theta_m\rangle\langle\theta_m| \tag{2}$$

with $\hat{\phi}_\theta|\theta_m\rangle = \theta_m|\theta_m\rangle$ [5]. In these equations $M+1$ stands for a Hilbert space dimension. In this scheme, the ideal PS will emerge in the limit $M\to\infty$. The properties of this state have been studied in [5, 6, 21], and a scheme for its generation was proposed in [22].

In this report we will present a distinct strategy to define the PS. Initially, we concentrate our attention on a *partial phase state*(**PPS)** [6], exhibiting a highly peaked phase distribution for a convenient choice of involved parameters. In this scenario an ideal PS emerges also from a limiting procedure, as in [5, 6], but here the approach is different: the limit is accomplished upon the parameters of the state itself, not upon the dimension of a Hilbert space as implemented in Ref.[5, 6].

This paper is arranged as follows: in the Section II we characterize our new PPS based on its generation and study some of its nonclassical properties. The Section III treats its degree of nonclassicality. In the Sect. IV we introduce the new *PS* and the Section V contains the comments and conclusion.

II. THE PARTIAL PHASE STATE

A. Definition

The general PPS is defined as [6],

$$|b\rangle = \sum_{n=0}^{s} b_n e^{in\theta}|n\rangle \tag{3}$$

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where b_n is real, positive. Examples of this state are given in [6]: first, the "rectangular" state $|b, r\rangle$ for which the coefficients b_n equal a constant $(r^{-1/2})$ for $q \leq n < q + r$ and vanishes elsewhere. The photon number distribution is constant and nonzero between $|q\rangle$ and $|q + r - 1\rangle$ and is zero outside this interval - hence the name "rectangular" state. Special cases of this state are the PS ($q = 0$ and $r = s + 1$) and the number state $|q\rangle$ ($r = 1$). Second, the physical PPS, of which the coherent state is a particular example. PPS viewed as a nonlinear coherent state [23] and as hypergeometric state [24] have been also considered in the literature.

Another example of physical PPS comes from a convenient choice of ideal squeezed state $|\alpha, z\rangle$, named *phase-squeezed state* [4]. An arbitrary ideal squeezed state is defined as $|\alpha, z\rangle = \hat{D}(\alpha)\hat{S}(z)|0\rangle$ where $\hat{D}(\alpha) = exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$ and $\hat{S}(z) = exp[(z^*\hat{a}^2 - z\hat{a}^{\dagger 2})/2]$ with $\alpha = Re^{i\theta}$ and $z = re^{i\phi}$. Its expansion in the Fock's basis, $|\alpha, z\rangle = \sum_{n=0}^{\infty} C_n|n\rangle$, has coeficients [3]

$$C_n = (2^n n!\mu)^{-\frac{1}{2}} \left(\frac{\nu}{\mu}\right)^{\frac{n}{2}} e^{-\frac{|\beta|^2}{2} + \frac{\nu^*}{2\mu}\beta^2} H_n\left(\frac{\beta}{\sqrt{2\mu\nu}}\right) \quad (4)$$

where $\mu = cosh(r)$, $\nu = e^{i\phi}sinh(r)$ and $\beta = \mu\alpha + \nu\alpha^*$. The *phase-squeezed state* emerges from the choice $\theta = (\phi + \pi)/2$ as shown in Fig.1. Although never being characterized in the literature as a PPS their phase properties were studied, based on the Pegg-Barnet formalism [25].

In this section we will identify the *phase-squeezed state* as a PPS. To this end we will show how to write the coefficients C_n of Eq.(4) in the form required by the PPS, given in Eq.(3). In the phase space, the ideal squeezed state is represented by an elipse centered at $\langle\hat{a}\rangle = \alpha$, having major (minor) axis e^r (e^{-r}), as shown in Fig.1. If we set $\theta = (\phi + \pi)/2$ and $|\alpha| = R = e^r$, the mentioned

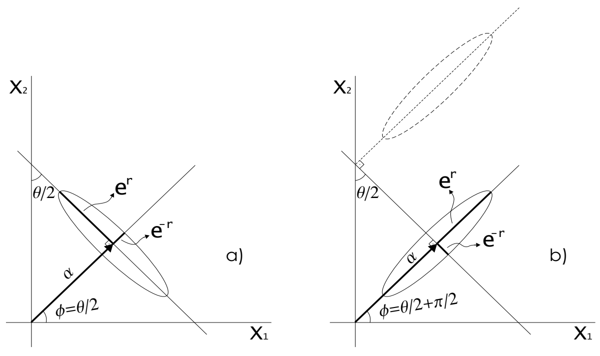

FIG. 1: Ideal Number (a) and Phase (b) Squeezed States

elipse touches the origin in the phase space and exhibits reduced phase dispersion as shown in Fig.2.

Now, we define the new PPS, denoted by $|R, \theta\rangle$, as an ideal squeezed state obeying the following prescriptions

$$\begin{aligned} \phi &= 2\theta - \pi\,, \\ r &= \ln R. \end{aligned} \quad (5)$$

Since $r > 0$ then R is restricted to $R > 1$. Replacing (5) in (4) results the coefficients,

$$\begin{aligned} C_n &= \sqrt{\frac{2R}{R^2+1}} \frac{e^{\frac{-R^2}{R^2+1}}}{\sqrt{n!}} \left(-i\sqrt{\frac{R^2-1}{2(R^2+1)}}\right)^n \\ &\times\ H_n\left(i\sqrt{\frac{2R^2}{R^4-1}}\right) e^{in\theta}, \end{aligned} \quad (6)$$

having the wanted form $C_n = b_n e^{in\theta}$ of the Eq.(3).

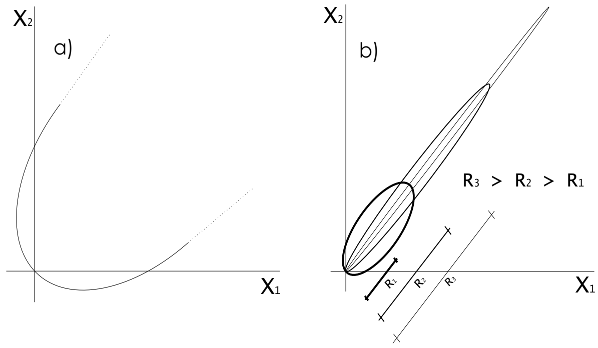

FIG. 2: Pictorial representation of the new PPS in the phase space.

Fig.2 shows plots of our PPS in the phase space, for various values of the parameter R. Fig.2a is a window of Fig.2b displaying an elipse near the origin. Note that the elipse in the 1^{st} quadrant crosses the 2^{nd} and 4^{th} quadrants of the phase space; this is an undesired result, since the elipse should remain on a single quadrant in order to characterize a single PPS. Notwithstanding, these undesired intersections tends to zero when R becomes large.

Concerning the generation of this new PPS, now it turns out immediate: it is obtained from a special class of squeezed states $|\alpha, z\rangle$, that obeying the prescriptions (5). Generation of arbitrary squeezed states for stationary fields in a high-Q cavity has been discussed recently [26]. For states of travelling fields, there are traditional schemes [27].

B. Nonclassical Properties

1. Photon Number Distribution

Using the Eq.(6), the photon number distribution for the PPS, $P_n = |C_n|^2$, results

$$\begin{aligned} P_n &= \frac{2R}{R^2+1} \frac{e^{-\frac{2R^2}{R^2+1}}}{2^n n!} \left(\frac{R^2-1}{R^2+1}\right)^n \\ &\times\ \left|H_n\left(i\sqrt{\frac{2R^2}{R^4-1}}\right)\right|^2. \end{aligned} \quad (7)$$

Note that P_n tends to zero when R increases without limit. Fig.3 displays plots of P_n versus n, for two values of R (and $r = \ln R$). It shows the variation of P_n for n even (blank) and n odd (shadow). Note that both envelops tends to the same limit. In other words, the limit of P_n for $n \to \infty$ lies between the even and odd envelops, a value determined by the parameter $R = |\alpha|$. When R (hence r) increases this limit goes to zero. In resume, for $n \gg 1$, P_n becomes almost constant. The same becomes true for small values of n when $R \gg 1$.

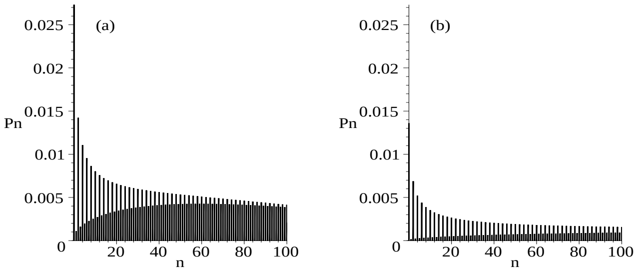

FIG. 3: Plots of P_n versus n for (a) $R = 10$; and (b) $R = 20$.

2. Atomic Inversion

In experiments involving electromagnetic cavities, one monitors the population of atomic states as function of time [28]. For the case of a two-level (Rydberg) atom interacting with a (single mode) field, with the whole system being previously prepared in a given state, the time evolution of the atom-field system is described by the Jaynes-Cummings Hamiltonian [29]. When one assumes resonance between the field and atom, plus the rotating wave aproximation [30], this Hamiltonian is written as

$$\hat{H} = \hbar w \hat{a}^\dagger \hat{a} + \hbar w \hat{\sigma}_z/2 + \hbar\lambda(\hat{\sigma}^+\hat{a} + \hat{\sigma}^-\hat{a}^\dagger) \tag{8}$$

where $\hat{a}^\dagger$ ($\hat{a}$) is the creation (annihilation) operator for photons; $\hat{\sigma}_z = (|e\rangle\langle e| - |g\rangle\langle g|)$ is the atomic inversion operator, with $|e\rangle$ ($|g\rangle$) standing for the atomic excited (ground) state; $\hat{\sigma}^+$ ($\hat{\sigma}^-$) is the raising (lowering) operator for the atom; w is the field (and atomic) frequency and λ stands for the atom-field coupling constant.

The atomic inversion is obtained [31] via $W(T) = \langle\Psi_{AF}(0)|\hat{\sigma}_z(T)|\Psi_{AF}(0)\rangle$, with $T = \lambda t$. In this expression $|\Psi_{AF}(0)\rangle$ stands for the initial state describing the whole atom-field system. Here we take $|\Psi_{AF}(0)\rangle = |\Psi_A(0)\rangle|\Psi_F(0)\rangle$ with $|\Psi_A(0)\rangle = |e\rangle$ for brevity and $|\Psi_F(0)\rangle$ as our new PPS. Then we find that: $W(T) = \sum_{n=0}^{\infty} P_n cos(2T\sqrt{n+1})$, with P_n given in Eq.(7).

Fig.4 shows the atomic inversion as function of time $T = \lambda t$, for (a) $R = 1$, and (b) $R = 10$. We note that this quantity exhibits no collapse and revival effect and its amplitude of oscillations tends to zero when R becomes large, an expected result when the phase of a state becomes improved [32].

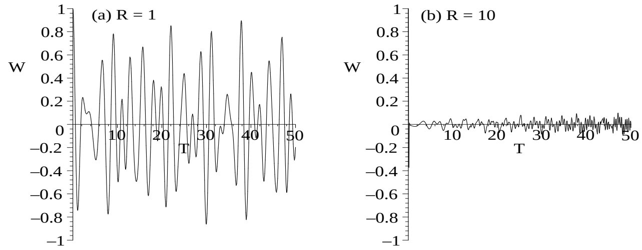

FIG. 4: Plots of atomic inversion W versus time $T = \lambda t$.

3. Quasi-Distributions

Using the characteristic function $\chi(\eta)$ for a pure state $|\Psi\rangle$, $\chi(\eta) = \langle\Psi|\hat{D}(\eta)|\Psi\rangle$, we obtain a general class of quasi-distributions [33, 34], written as

$$F(\zeta; w) = \frac{1}{\pi^2}\int_{all} e^{-\frac{w}{2}|\eta|^2 + \eta^*\zeta - \eta\zeta^*}\chi(\eta)d^2\eta. \tag{9}$$

For $w = 0$, $F(\zeta; 0)$ furnishes the Wigner distribution, while $w = 1$ yields the Husimi Q representation, $Q = F(\zeta; 1)$. For our PPS the Eq.(8) results, with $\zeta = x + iy$,

$$F(x, y; w) = \frac{(b^2 - 4|c|^2)^{\frac{-1}{2}}}{\pi} exp\left(\frac{2Re(a^2c)b|a|^2}{b^2 - 4|c|^2}\right) \tag{10}$$

where $a = Re^{-i\theta} - x + iy$, $b = [(R^2+1)/2R]^2 + (w-1)/2$ and $c = e^{2i\theta}(R^4 - 1)/8R^2$. The greatest value of this function occurs at $x = Rcos(\theta)$ and $y = Rsin(\theta)$ when $a = 0$. Then the Eq.(9) gives

$$F_{max}(x, y; w) = \left(\pi\sqrt{b^2 - 4|c|^2}\right)^{-1}, \tag{11}$$

yelding $W_{max} = 2/\pi$ for $w = 0$. For $w = 1$, we obtain

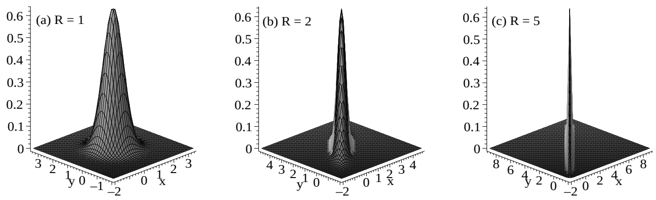

FIG. 5: Plots of Wigner function, for (a) $R = 1$; (b) $R = 2$ and (c) $R = 5$.

$Q_{max} = 2R/[\pi(R^2 + 1)]$ and this expression goes to 0 when R grows. From the prescription (5) $R = 1$ implies $r = 0$, concerned with a coherent state. In this case $Q_{max} = 1/\pi$, as it should [33]. This Q-function will be useful in the Sec.III. Fig.(5) displays the Wigner function for $R = 1$ (a); $R = 2$ (b) and $R = 5$ (c). As wanted, it shows the phase definition of our PPS being improved when R (and r) grows.

4. Phase Distribution

There are various theoretical approaches describing the phase observable. Some of them are motivated by the aim of expressing the phase as the complement of the photon number, in the spirit of Dirac's original work [35]. Although these approaches are quite distinct, they all lead to the same phase probability distribution for a field in state $|\Psi\rangle$ as a function of phase angle Θ [36]:

$$P(\Theta) = \frac{1}{2\pi}\left|\sum_{n=0}^{\infty}\langle n|\Psi\rangle e^{-in\Theta}\right|^2. \tag{12}$$

Replacing in (12) the $C_n = \langle n|\Psi\rangle$ given in (6) we find the phase distribution for our PPS,

$$\begin{aligned} P(\Theta) &= \frac{1}{\pi}\frac{Re^{\frac{-2R^2}{R^2+1}}}{R^2+1}\left|\sum_{n=0}^{\infty}\left(-i\sqrt{\frac{R^2-1}{2(R^2+1)}}\right)^n \right. \\ &\times \left. H_n\left(i\sqrt{\frac{2R^2}{R^4-1}}\right)\frac{e^{in(\theta-\Theta)}}{\sqrt{n!}}\right|^2. \end{aligned} \tag{13}$$

Fig.(6) shows plots of $P(\Theta)$ for $\theta = \pi/2$. Note that the distributions are very concentrated and simetric around $\Theta = \pi/2$ and, when R grows, $P(\Theta)$ becomes more concentrated until reaching a Dirac distribution $\delta(\theta - \Theta)$.

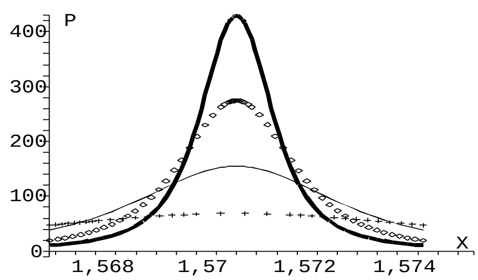

FIG. 6: $P(\Theta)$ versus $x = \Theta$ near $\Theta = \pi/2$ for $R = 10$ (cross), $R = 15$ (normal), $R = 20$ (circle) and $R = 25$ (hard).

This reminds us of the Pegg-Barnett *truncated* PS reaching the Pegg-Barnett PS when the dimension of a Hilbert space grows [5].

III. NONCLASSICAL DEPTH OF THE NEW PPS

A pertinent question in quantum optics is: "given two states exhibiting distinct nonclassical effects, which of them is more nonclassical than the other ? " . This question has been addressed by many authors in the literature [37]. More recently [38], inspired on [39], the issue has been considered analysing both phase-space and distance-type measures of nonclassicality. In [38] the criterium to quantify the nonclassical depht of a state $|\Psi\rangle$ is taken as the minimum of distance of this state and that of the nearest classical (coherent) state. A straighforward procedure using this concept leads to the operational formula,

$$d_m(\Psi) = 1 - \pi Q_{max}(\Psi) \tag{14}$$

where $d_m(\Psi)$ stands for the mentioned minimum distance (nonclassical depth) and $Q_{max}(\Psi)$ stands for maximum of Q-Husimi function. For our PPS the Eq.(13) results

$$d_m = \frac{(R-1)^2}{R^2+1}. \tag{15}$$

Remembering that $R \in [1,\infty)$, the Eq.(14) yields $d_m \to 0$ (most classical) if $R \to 1$, whereas $d_m \to 1$ (most nonclassical) if $R \to \infty$. These results can also be obtained via the Lee criterium since it works for Gaussian states. These results give us a route to achieve a new PS, as considered in the following section.

IV. THE PPS AS A GUIDE TO A NEW PS

All previous considerations stand as a support justifying the introduction of a new PS. In the present scenario it emerges naturally, as the limit of our PPS,

$$|\theta\rangle = \lim_{R\to\infty}|R,\theta\rangle. \tag{16}$$

These states are (Dirac) delta-orthonormalized: $\langle\theta|\theta'\rangle = \delta(\theta' - \theta)$. To show this we take

$$\langle\theta|\theta'\rangle = \lim_{R,R'\to\infty}\langle R,\theta|R',\theta'\rangle = \lim_{R,R'\to\infty}\sum_{n=0}^{\infty}C_n^* C_n', \tag{17}$$

where $\sum_{n=0}^{\infty}|n\rangle\langle n| = \hat{1}$ has been used and C_n are given in Eq.(6). Next, using the Mehler identity [40]

$$\sum_{n=0}^{\infty}\frac{z^n H_n(x)H_n(y)}{n!2^n(1-z^2)^{-1/2}} = exp\left(\frac{2xyz-(x^2+y^2)z^2}{1-z^2}\right) \tag{18}$$

we obtain

$$\langle R,\theta|R',\theta'\rangle = \frac{f(R)f(R')}{\sqrt{1-z^2}}exp\left(\frac{2xyz-(x^2+y^2)z^2}{1-z^2}\right). \tag{19}$$

where $x = i\sqrt{2R^2/(R^4-1)}$, $y = -i\sqrt{2R'^2/(R'^2-1)}$, $z = \sqrt{(R^2-1)(R'^2-1)/[(R^2+1)(R'^2+1)]}exp[i(\theta-\theta')]$ and $f(R) = \sqrt{2R/(R^2+1)}e^{-R^2/(R^2+1)}$. Pursuing further this line and neglecting higher order terms ($1/R^2$, $1/R'^2$, etc) in comparison with $1/R$ and $1/R'$ we obtain from Eq.(18): $\langle\theta|\theta'\rangle = \delta(\theta-\theta')$, for $(\theta-\theta') \in (-\pi,+\pi)$ [41].

Finally, and inspired on [5], the corresponding phase operator $\hat{\Phi}$ is defined in terms of a suitable PS basis, as follows,

$$\hat{\Phi} = \int_0^{2\pi} \theta|\theta\rangle\langle\theta| d\theta \tag{20}$$

yielding the eigenvalue equation,

$$\hat{\Phi}|\theta\rangle = \theta|\theta\rangle. \tag{21}$$

It is worth stressing that in the present context the operator $\hat{\Phi}$ in Eqs.(20), (21) make sense only when applying the limit $R \rightarrow \infty$ (as used in Eq. (17)), this limit taken after all calculations of expectation values and similar c-numbers. This is a feature in common with the Pegg-Barnett approach [5] and seems to be inevitable.

V. COMMENTS AND CONCLUSION

In this report we have considered the properties and engineering of new states of the quantized electromagnetic field. In this scenario we have focused the class of PPS (Sect.II), studing a new kind of such state, some of its representative properties (Sect.III) and its nonclassical depth (Sec.IV). As a natural consequence, the results found in the Sects. II and III support the introduction of a new kind of PS: it emerges from a suitable limit of the new PPS for R (hence r) $\rightarrow \infty$. Contrary to the strategy used in [5, 6], where the limiting process is taken on the Hilbert space dimension, $N \rightarrow \infty$, here the limit is implemented upon the arrangement of a coherent state ($|\alpha| = R \rightarrow$ large) plus a radial and strong squeezing in the phase space, all obeying the prescription in Eq.(5) to avoid appearance of spurious bifurcations in the phase space, as found in [41]. Of course, reaching the limit R (hence r)$\rightarrow \infty$ is impossible experimentally, since this requires infinite energy [42]. So, our PS defined in Eq.(15) is an *ideal state* - in the same way as, e.g., a plane wave describing a state having well defined momentum, also corresponds to an *ideal state.*

As final remarks, we mention previous works involving a phase state [43]. The present PS has some analogies with those of [43], but there is a crucial distinction: the use of our Eq.(5), which realizes the idea by Vogel-Schleich [43]- that a 'true' PS should correspond to a ray starting from the origin of the phase space. Further investigations on the properties of these new PPS and PS, their comparison with those found in [5,6], are in progress and will be considered in a future work.

VI. ACKNOWLEDGMENTS

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